The Seiberg–Witten map for non-commutative pure gravity and vacuum Maxwell theory

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Abstract

In this paper the Seiberg–Witten map is first analyzed for non-commutative Yang–Mills Theories with the related methods, developed in the literature, for its explicit construction, that hold for any gauge group. These are exploited to write down the second-order Seiberg–Witten map for pure gravity with a constant non-commutativity tensor. In the analysis of pure gravity when the classical space-time solves the vacuum Einstein equations, we find for three distinct vacuum solutions that the corresponding non-commutative field equations do not have solution to first order in non-commutativity, when the Seiberg–Witten map is eventually inserted. In the attempt of understanding whether or not this is a peculiar property of gravity, in the second part of the paper, the Seiberg–Witten map is considered in the simpler case of Maxwell theory in vacuum in the absence of charges and currents. Once more, no obvious solution of the non-commutative field equations is found, unless the electromagnetic potential depends in a very special way on the wave vector.

I. INTRODUCTION

Field theories on a non-commutative space can be obtained by replacing the ordinary products with the $\star$-product. This idea has been widely developed over the last years, mainly because the non-commutative gauge theory turns out to be a limit of string theory [1]. This relation with string theory makes it possible to map non-commutative theories into commutative ones [1] with the help of the Seiberg–Witten map which is a gauge equivalence relation between non-commutative gauge theory and its ordinary counterpart which is compatible with the gauge structure of the theory. The Seiberg–Witten map for non-commutative fields can be written for arbitrary non-Abelian gauge groups. Explicit Seiberg–Witten maps of non-commutative fields are useful both to understand physical predictions and to check the behaviour of non-commutative theory itself, e.g. perturbative renormalizability.

The Seiberg–Witten maps for gauge parameter and fields can be obtained as solutions of consistency conditions of the gauge theory under consideration [2], that are analyzed perturbatively by expanding gauge parameter and fields in a formal power series in the non-commutativity parameter θ. This method has provided explicit solutions for the Seiberg–Witten map of non-Abelian gauge theories only up to the second order in θ [2–7].

In the work in Ref. [8], the Seiberg–Witten map for gauge parameter, gauge field and matter fields is written to all orders for a non-commutative non-Abelian theory. The authors of Ref. [8] show that the map to second order for the fields [4] can be written in terms of their counterparts to zeroth and first order in the expansion of the map. By virtue of this structure of the second-order calculation, in Ref. [8] a recursive formula has been built for all orders that satisfies the Seiberg–Witten map.

By exploiting this correspondence, in the present paper we build the second-order Seiberg–Witten map for a theory of pure gravity expressed in the tetrad formalism. By doing so we introduce a set of local Lorentz frames, whose global existence is ensured if the classical space-time manifold is parallelizable, and we regard gravity as a gauge theory on a non-commutative space defined by

$$[x^\mu, x^\nu]_\star = \mathrm{i}\theta^{\mu\nu} \ , \tag{1.1}$$

where θ is a constant Poisson tensor, and the $\star$-product is the associative Weyl–Moyal product

$$f \star g = f \, \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{jk}\overleftarrow{\partial_j}\overrightarrow{\partial_k}} g. \tag{1.2}$$

Here we exploit the fact that the derivative of functions satisfies the Leibniz rule with respect to the $\star$-product, i.e.

$$\partial_i(f \star g) = (\partial_i f) \star g + f \star (\partial_i g), \tag{1.3}$$

as in the case of the ordinary product; this imples that θ is constant.

In light of all these considerations, in Sec. II we review the fundamental concepts for building a Seiberg–Witten map for a Yang–Mills theory, and how is it built to first order. In Sec. III, we summarize the strategies for generalizing the Seiberg–Witten map adopted in Refs. [8] and [10]. In Sec. IV, following the work in Ref. [8], we apply the recursive procedure for providing the second-order Seiberg–Witten map for the tetrad, which can be written in terms of fields to zeroth order in the non-commutativity parameter, and in terms of first-order solutions. Section V writes the pure-gravity action to second order in non-commutativity. In Sec. VI we consider pure gravity when the classical space-time solves the vacuum Einstein equations (three distinct cases). The resulting non-commutative field equations are found not to have solution to first order in non-commutativity, when the Seiberg–Witten map is eventually inserted. In the attempt of understanding whether or not this is a peculiar property of gravity, Sec. VII studies the simpler case of vacuum Maxwell theory in the absence of charges and currents. Concluding remarks and open problems are presented in Sec. VIII, while the Appendix describes the attempt of solving the non-commutative field equations of pure gravity without using the Seiberg–Witten map.

II. THE SEIBERG–WITTEN MAP TO FIRST ORDER

For a Yang–Mills theory, the gauge potential is a Lie-algebra-valued 1-form. Following the notation in Ref. [1], the gauge transformations for components of potential and field strength read as

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda - \mathrm{i}[A_\mu, \Lambda] \equiv D_\mu \Lambda, \quad \delta_\Lambda F_{\mu\nu} = \mathrm{i}[\Lambda, F_{\mu\nu}], \tag{2.1}$$

where the Lie-algebra indices (frequently written upstairs and taken from the beginning of the Greek alphabet) are omitted for simplicity. With this understanding, one can write the Yang–Mills field strength in the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu], \tag{2.2}$$

out of which one can get the gauge curvature 2-form

$$F = F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{2.3}$$

For a non-commutative Yang–Mills theory, one uses the same formulae for gauge transformation and field strength, with the understanding that the ordinary product of matrices is replaced by the Moyal $\star$-product [1], i.e.

$$\hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu = \partial_\mu \hat{\Lambda} - \mathrm{i}[\hat{A}_\mu, \hat{\Lambda}]_\star \equiv \hat{D}_\mu \hat{\Lambda}, \quad \hat{\delta}_{\hat{\Lambda}} \hat{F}_{\mu\nu} = \mathrm{i}[\hat{\Lambda}, \hat{F}_{\mu\nu}]_\star, \tag{2.4}$$

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}[\hat{A}_\mu, \hat{A}_\nu]_\star \tag{2.5}$$

is the non-commutative field strength of the non-commutative gauge potential $\hat{A}$. The resulting theory reduces to the familiar Yang–Mills theory with group $U(N)$ as $\theta \to 0$.

The Seiberg–Witten map relates non-commutative fields and their commutative counterparts through a gauge equivalence relation [1]:

$$\hat{A}_\mu(A;\theta) + \hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu(A;\theta) = \hat{A}_\mu(A + \delta_\Lambda A; \theta), \tag{2.6}$$

where A and Λ are the commutative gauge field (or potential) and gauge parameter, respectively.

Equation (2.6) can be re-written in the form

$$\hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu(A,\theta) = \hat{A}_\mu(A + \delta_\Lambda A;\theta) - \hat{A}_\mu(A;\theta) = \delta_\Lambda \hat{A}_\mu(A;\theta). \tag{2.7}$$

To first order in θ, denoted by the superscript 1, the formulae (2.4) become

$$\delta_\Lambda A_i^1[a] = A_\Lambda^1(A+\delta_\Lambda A) - A_\Lambda^1(A) = \partial_i \Lambda^1[A] + \mathrm{i}[\Lambda^1[A], A_i] + \mathrm{i}[\Lambda, A_i^1[a]] - \frac{1}{2}\theta^{kl}\{\partial_k \Lambda, \partial_l A_i\}. \tag{2.8}$$

The solution of Eq. (2.8) to first order in θ is (see Eq. (3.5) of [1])

$$A_\gamma^1 = -\frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa, \partial_\lambda A_\gamma + F_{\lambda\gamma}\}, \quad \Lambda^1 = \frac{1}{4}\theta^{\kappa\lambda}\{\partial_\kappa \Lambda, A_\lambda\}. \tag{2.9}$$

We can check it in particular for the gauge field, and analogous procedure holds for the gauge parameter and field strength. By applying the gauge transformation (2.4) to the first of Eqs. (2.9), and applying again Eq. (2.4) within it, one finds

$$\delta_\Lambda A_\gamma^1[a] \;=\; -\frac{1}{4}\theta^{\kappa\lambda}\{\delta_\Lambda A_\kappa, \partial_\lambda A_\gamma + F_{\lambda\gamma}\} - \frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa, \partial_\lambda \delta_\Lambda A_\gamma + \delta_\Lambda F_{\lambda\gamma}\}$$

$$
\begin{aligned}
&- \frac{1}{4}\theta^{\kappa\lambda}\{\partial_\kappa\Lambda - \mathrm{i}[A_\kappa,\Lambda], \partial_\lambda A_\gamma + F_{\lambda\gamma}\} - \frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa, \partial_\lambda\partial_\gamma\Lambda\} \\
&+ \frac{\mathrm{i}}{4}\theta^{\kappa\lambda}\{A_\kappa, [\partial_\lambda A_\gamma, \Lambda]\} + \frac{\mathrm{i}}{4}\theta^{\kappa\lambda}\{A_\kappa, [A_\gamma, \partial_\lambda\Lambda]\} \\
&- \frac{\mathrm{i}}{4}\theta^{\kappa\lambda}\{A_\kappa, [\Lambda, F_{\lambda\gamma}]\}. \qquad (2.10)
\end{aligned}
$$

On the other hand, by inserting Eqs. (2.9) into the right-hand side of Eq. (2.8), one finds

$$
\begin{aligned}
\delta_\Lambda A^1_\gamma[a] \;=\; &\frac{1}{4}\theta^{\kappa\lambda}\{\partial_\gamma\partial_\kappa\Lambda, A_\lambda\} + \frac{1}{4}\theta^{\kappa\lambda}\{\partial_\kappa\Lambda, \partial_\gamma A_\lambda\} + \frac{\mathrm{i}}{4}\theta^{\kappa\lambda}[\{\partial_\kappa\Lambda, A_\lambda\}, A_\gamma] \\
&- \frac{\mathrm{i}}{4}\theta^{\kappa\lambda}[\Lambda, \{A_\kappa, \partial_\lambda A_\gamma + F_{\lambda\gamma}\}] - \frac{1}{2}\theta^{\kappa\lambda}\{\partial_\kappa\Lambda, \partial_\lambda A_\gamma\}. \qquad (2.11)
\end{aligned}
$$

By comparing Eq. (2.10) with Eq. (2.11), and using the following property:

$$
\{A, [B, C]\} + \{B, [A, C]\} = [\{A, B\}, C], \qquad (2.12)
$$

one checks the equality and hence Eq. (2.8) is solved by (2.9).

III. THE SEIBERG–WITTEN MAP TO ALL ORDERS

In this section we are going to summarize the various ways available for building a Seiberg–Witten map to all orders in θ, in particular those proposed in Refs. [8], [9], [10].

First way.

In Ref. [8] the authors consider, in order to build a Seiberg–Witten map to all orders, the following series expansion in θ:

$$
\begin{aligned}
\hat{\Lambda}_\alpha \;&=\; \alpha + \Lambda^1_\alpha + \cdots + \Lambda^n_\alpha + \cdots, \qquad (3.1) \\
\hat{A}_\mu \;&=\; A_\mu + A^1_\mu + \cdots + A^n_\mu + \cdots,
\end{aligned}
$$

where the zeroth order terms α and A_μ are the commutative counterparts of $\hat{\Lambda}_\alpha$ and $\hat{A}_\mu$. Moreover, one denotes by $\star^r$ the r-th order term in the expansion of the $\star$-product, i.e.

$$
f(x)\star^r g(x) = \frac{1}{r!}\left(\frac{\mathrm{i}}{2}\right)^r \theta^{\mu_1\nu_1}\cdots\theta^{\mu_r\nu_r}\partial_{\mu_1}\cdots\partial_{\mu_r}f(x)\partial_{\nu_1}\cdots\partial_{\nu_r}g(x). \qquad (3.2)
$$

By exploiting the first-order Seiberg–Witten map given in (2.9), one can obtain its second-order form from the Eqs. [8]

$$
\Lambda^2_\alpha = -\frac{1}{8}\theta^{\kappa\lambda}\left(\{A^1_\kappa, \partial_\lambda\alpha\} + \{A_\kappa, \partial_\lambda\Lambda^1_\alpha\}\right) - \frac{\mathrm{i}}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}[\partial_\mu A_\kappa, \partial_\nu\partial_\lambda\alpha], \qquad (3.3)
$$

$$
\begin{aligned}
A^2_\gamma &= -\frac{1}{8}\theta^{\kappa\lambda}\left(\{A^1_\kappa,\partial_\lambda A_\gamma + F_{\lambda\gamma}\} + \{A_\kappa,\partial_\lambda A^1_\gamma + F^1_{\lambda\gamma}\}\right) \\
&- \frac{\mathrm{i}}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}[\partial_\mu A_\kappa,\partial_\nu(\partial_\lambda A_\gamma + F_{\lambda\gamma})],
\end{aligned} \tag{3.4}
$$

and the authors of Ref. [8] stress that terms linear in θ in the formulae above result from an expansion where the first-order terms are given by (2.9), and hence quadratic terms are related to the expansion of the $\star$-product in powers of θ. Thus, by inspection of the structure of the two expansions to first and second order, respectively, the authors [8] make the following conjecture for the general structure of the Seiberg–Witten map to all orders in θ, i.e. a recursive formula to all orders reading as

$$
\Lambda^{n+1}_\alpha = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\sum_{p+q+r=n}\{A^p_\kappa,\partial_\lambda\Lambda^q_\alpha\}_{\star^r}, \tag{3.5}
$$

$$
A^{n+1}_\gamma = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\sum_{p+q+r=n}\{A^p_\kappa,\partial_\lambda A^q_\gamma + F^q_{\lambda\gamma}\}_{\star^r}, \tag{3.6}
$$

where the first-order solution is obtained by setting $n = 0$, the second-order solution by setting $n = 1$, and so on.

Second way.

There exists an alternative approach to studying the Seiberg–Witten map, that relies upon finding the same solution starting from a differential equation introduced in [1], whose solution is given by

$$
\Lambda^{n+1}_\alpha = -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\cdots\partial\theta^{\mu_n\nu_n}}\sum_{p+q+r=n}\{A^p_\mu,\partial_\nu\Lambda^q_\alpha\}_{\star^r}\right)_{\theta=0}, \tag{3.7}
$$

$$
A^{n+1}_\gamma = -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\cdots\partial\theta^{\mu_n\nu_n}}\{\hat{A}^{(n)}_{\mu_1},\partial_{\nu_1}\hat{A}^{(n)}_\gamma + \hat{F}^{(n)}_{\nu_1\gamma}\}_\star\right)_{\theta=0} \tag{3.8}
$$

Equations (3.7), (3.8) reduce to the previous recursive formulae (3.5), (3.6).

Third way

Yet another approach to studying the Seiberg–Witten map relies upon a differential equation of time-evolution type [10]. The authors of this paper introduce a time parameter t such that $\theta^{ij} \to t\,\theta^{ij}$, $\Lambda \to \Lambda(t)$ and $A_i \to A_i(t)$. In such a way, Λ e A_i acquire a time dependence through θ. The evolution equation provides a method useful for the evaluation

of higher-order terms by differentiation with respect to t. One can then compute $A_i^{(n)}$ and $\Lambda^{(n)}$ according to

$$\Lambda^{(n)} = \frac{1}{n!}\frac{\partial^n \Lambda(t)}{\partial t^n}\,|_{t=0}, \qquad A_i^{(n)} = \frac{1}{n!}\frac{\partial^n A_i(t)}{\partial t^n}\,|_{t=0}\ . \tag{3.9}$$

There exist other techniques for finding the Seiberg–Witten map at higher orders, based upon the homotopy operator, applied order by order, as suggested in Refs. [11, 12].

IV. SEIBERG–WITTEN MAP FOR PURE GRAVITY

In this section we relate a non-commutative version of pure gravity to its commutative counterpart by using the Seiberg–Witten map, relying upon the work in Ref. [13]. In this case the Seiberg–Witten map relates the non-commutative degrees of freedom $\hat{\omega}_\mu$, $\hat{e}_\mu$ and $\hat{\Lambda}$ to their commutative counterparts, i.e. classical spin connection ω_μ, tetrad e_μ and gauge parameter Λ.

The map between $\hat{\omega}_\mu$ and ω_μ, and between $\hat{e}_\mu$ and e_μ to first order in the constant non-commutativity $\theta^{\mu\nu}$ is given by [13]

$$\hat{e}_\mu = e_\mu - \frac{1}{2}\theta^{\lambda\sigma}\{\omega_\lambda, \partial_\sigma e_\mu + \frac{\mathrm{i}}{2}[e_\mu, \omega_\sigma]\}, \tag{4.1}$$

$$\hat{\omega}_\mu = \omega_\mu - \frac{1}{4}\theta^{\lambda\sigma}\{\omega_\lambda, \partial_\sigma \omega_\mu + R_{\sigma\mu}\}. \tag{4.2}$$

We now perform in the first way, as seen in the previous Section, the construction of the Seiberg–Witten map for pure gravity with the understanding that $A_\kappa^p = \omega_\kappa^p$ e $F_{\lambda\gamma}^q = R_{\lambda\gamma}^q$:

$$\hat{e}_\mu^{n+1} = -\frac{1}{2(n+1)}\theta^{k\lambda}\left(\sum_{p+q+r=n}\{\omega_\kappa^p, \partial_\lambda e_\mu^q\}_{\star^r} + \frac{\mathrm{i}}{2}\sum_{p+q+r+s=n}\{\omega_k^p, [e_\mu^q, \omega_\lambda^s]\}_{\star^r}\right). \tag{4.3}$$

In particular, for the second-order map for the spin-connection one now finds (this order implies setting $n = 1$)

$$\begin{aligned} \omega_\gamma^2 \ = \ & -\frac{1}{8}\theta^{\kappa\lambda}\left(\{\omega_\kappa^1, \partial_\lambda\omega_\gamma + R_{\lambda\gamma}\} + \{\omega_\kappa, \partial_\lambda\omega_\gamma^1 + R_{\lambda\gamma}^1\}\right) \\ & -\frac{\mathrm{i}}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}[\partial_\mu\omega_\kappa, \partial_\nu(\partial_\lambda\omega_\gamma + R_{\lambda\gamma})] \end{aligned} \tag{4.4}$$

while for the tetrad

$$\begin{aligned} e_\mu^2 \ = \ & -\frac{1}{4}\theta^{\kappa\lambda}\left(\{\omega_\kappa^1, \partial_\lambda e_\mu\} + \{\omega_\kappa, \partial_\lambda e_\mu^1\}\right) - \frac{\mathrm{i}}{8}\theta^{k\lambda}\theta^{\sigma\nu}[\partial_\sigma\omega_k, \partial_\nu\partial_\lambda e_\mu] \\ & -\frac{\mathrm{i}}{8}\theta^{k\lambda}\left(\{\omega_k^1, [e_\mu, \omega_\lambda]\} + \{\omega_k, [e_\mu^1, \omega_\lambda]\} + \{\omega_\lambda, [e_\mu, \omega_\lambda^1]\}\right) \\ & +\frac{1}{16}\theta^{k\lambda}\theta^{\sigma\nu}\left([\partial_\sigma\omega_k, [\partial_\nu e_\mu, \omega_\lambda]] + [\partial_\sigma\omega_k, [e_\mu, \partial_\nu\omega_\lambda]] + \{\omega_k, \{\partial_\sigma e_\mu, \partial_\nu\omega_\lambda\}\}\right). \end{aligned} \tag{4.5}$$

From now on, with our notation, V is the tetrad 1-form expanded on the basis of Dirac γ-matrices, i.e.

$$V = V_\mu \mathrm{d}x^\mu = (V_\mu^a \gamma_a + \widetilde{V}_\mu^a \gamma_a \gamma_5)\mathrm{d}x^\mu = \hat{e}_\mu \mathrm{d}x^\mu = (\hat{e}_\mu^{(0)a}\gamma_a + \hat{e}_{\mu 5}^{(1)a}\gamma_5\gamma_a)\mathrm{d}x^\mu, \tag{4.6}$$

while R is the curvature 2-form

$$R = d\Omega - \Omega \wedge_* \Omega. \tag{4.7}$$

Our convention for the Minkowski metric is $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$, and we define $\sigma_{ab} \equiv -\frac{i}{4}[\gamma_a, \gamma_b] = -\frac{i}{2}\gamma_{ab}$. One thus finds

$$\hat{\omega}_\mu = \frac{1}{2}\hat{\omega}_\mu^{ab}\sigma_{ab} + \hat{a}_\mu + \mathrm{i}\hat{b}_{\mu 5}\gamma_5, \tag{4.8}$$

$$\hat{R}_{\rho\sigma} = \frac{\mathrm{i}}{2}\hat{R}_{\rho\sigma}^{ab}\sigma_{ab} + \mathrm{i}\hat{r}_{\rho\sigma} + \hat{\tilde{r}}_{\rho\sigma}\gamma_5. \tag{4.9}$$

Hereafter we use the following properties of γ-matrices:

$$\gamma_5\sigma_{ab} = \sigma_{ab}\gamma_5 = \frac{\mathrm{i}}{2}\epsilon_{abcd}\sigma^{cd}, \tag{4.10}$$

$$\{\sigma_{ab}, \gamma_5\} = \mathrm{i}\epsilon_{abcd}\sigma^{cd}, \tag{4.11}$$

$$[\sigma_{ab}, \gamma_5] = 0, \tag{4.12}$$

$$[\sigma_{ab}, \gamma_c] = \mathrm{i}(\eta_{ac}\gamma_b - \eta_{bc}\gamma_a), \tag{4.13}$$

$$\{\sigma_{ab}, \gamma_c\} = -\epsilon_{abc}{}^{d}\gamma_5\gamma_d, \tag{4.14}$$

$$\{\sigma_{ab}, \sigma_{cd}\} = \frac{1}{2}(\mathrm{i}\epsilon_{abcd}\gamma_5 + \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}), \tag{4.15}$$

from which

$$[\sigma_{ab}, \sigma_{cd}] = \mathrm{i}(\eta_{ac}\sigma_{bd} - \eta_{bc}\sigma_{ad} - \eta_{ad}\sigma_{bc} + \eta_{bd}\sigma_{ac}), \tag{4.16}$$

$$[\gamma_5\gamma_e, \sigma_{ab}] = \mathrm{i}\gamma_5(\eta_{be}\gamma_a - \eta_{ae}\gamma_b), \tag{4.17}$$

$$\{[\gamma_5\gamma_e, \sigma_{ab}], \sigma_{cd}\} = \mathrm{i}(\eta_{ae}\epsilon_{cdb}{}^{f} - \eta_{be}\epsilon_{cda}{}^{f})\gamma_f, \tag{4.18}$$

$$\{[\gamma_c, \gamma_5], \sigma_{ab}\} = 2\{\gamma_c\gamma_5, \sigma_{ab}\} = 2\epsilon_{abc}{}^{d}\gamma_d, \tag{4.19}$$

$$\{[\gamma_c, \sigma_{ab}], \gamma_5\} = 0, \tag{4.20}$$

$$\{\sigma_{ab}, \gamma_c\gamma_5\} = \epsilon_{abc}{}^{d}\gamma_d, \tag{4.21}$$

$$\{\sigma_{ab}, \{\gamma_c, \sigma_{ef}\}\} = -\epsilon_{efc}{}^{d}\epsilon_{abmd}\gamma_m, \tag{4.22}$$

$$[\sigma_{ab}, [\gamma_c, \sigma_{ef}]] = (\eta_{ec}\eta_{af} - \eta_{fc}\eta_{ae})\gamma_b + (\eta_{fc}\eta_{be} - \eta_{ec}\eta_{bf})\gamma_a, \tag{4.23}$$

and hence, by using (4.5) and (7.14)

$$\begin{aligned} e_\mu^2 &= -\frac{1}{4}\theta^{\kappa\lambda}\left(\{\omega_\kappa^1,\partial_\lambda e_\mu\}+\{\omega_\kappa,\partial_\lambda e_\mu^1\}\right)-\frac{\mathrm{i}}{8}\theta^{\kappa\lambda}\theta^{\sigma\nu}[\partial_\sigma\omega_\kappa,\partial_\nu\partial_\lambda e_\mu] \\ &-\frac{\mathrm{i}}{8}\theta^{\kappa\lambda}\left(\{\omega_\kappa^1,[e_\mu,\omega_\lambda]\}+\{\omega_\kappa,[e_\mu^1,\omega_\lambda]+\{\omega_\kappa,[e_\mu,\omega_\lambda^1]\}\}\right) \\ &+\frac{1}{16}\theta^{\kappa\lambda}\theta^{\sigma\nu}\left([\partial_\sigma\omega_\kappa,[\partial_\nu e_\mu,\omega_\lambda]]+[\partial_\sigma\omega_\kappa,[e_\mu,\partial_\nu\omega_\lambda]]+\{\omega_\kappa,\{\partial_\sigma e_\mu,\partial_\nu\omega_\lambda\}\}\right), \end{aligned} \quad (4.24)$$

where $\omega_\mu^1 = \hat{a}_\mu^{(1)} + \mathrm{i}\hat{b}_{\mu5}^{(1)}\gamma_5$, $e_\mu^1 = \hat{e}_{\mu5}^{(1)a}\gamma_5\gamma_a$, while $\omega_\mu = \frac{1}{2}\hat{\omega}_\mu^{(0)ab}\sigma_{ab}$, $e_\mu = \hat{e}_\mu^{(0)a}\gamma_a$. Thus, by substitution, a careful calculation leads eventually to (cf. Ref. [14])

$$\begin{aligned} e_\mu^2 &= -\frac{1}{4}\theta^{\kappa\lambda}\left(2\hat{a}_\kappa^{(1)}\partial_\lambda\hat{e}_\mu^{(0)a}\gamma_a-\frac{1}{2}\hat{\omega}_\kappa^{(0)ab}\partial_\lambda\hat{e}_{\mu5}^{(1)c}\epsilon_{abc}{}^d\gamma_d\right) \\ &+\frac{1}{16}\theta^{\kappa\lambda}\theta^{\sigma\nu}\partial_\sigma\hat{\omega}_\kappa^{(0)ab}\partial_\nu\partial_\lambda\hat{e}_\mu^{(0)c}(\eta_{ac}\gamma_b-\eta_{bc}\gamma_a) \\ &-\frac{1}{8}\theta^{\kappa\lambda}\left(\hat{a}_\kappa^{(1)}\hat{e}_\mu^{(0)c}\hat{\omega}_\lambda^{(0)ab}(\eta_{ac}\gamma_b-\eta_{bc}\gamma_a)\right. \\ &-\frac{1}{4}\hat{\omega}_\kappa^{(0)ab}\hat{e}_{\mu5}^{(1)c}\hat{\omega}_\lambda^{(0)de}(\eta_{dc}\epsilon_{abe}{}^f-\eta_{ec}\epsilon_{abd}{}^f)\gamma_f \\ &\left.-\hat{\omega}_\kappa^{(0)ab}\hat{e}_\mu^{(0)c}\hat{b}_{\lambda5}^{(1)}\epsilon_{abc}{}^d\gamma_d\right) \\ &+\frac{1}{64}\theta^{\kappa\lambda}\theta^{\rho\nu}\times\Big[\left(\partial_\rho\hat{\omega}_\kappa^{(0)ab}\partial_\nu\hat{e}_\mu^{(0)c}\hat{\omega}_\lambda^{(0)de}+\partial_\rho\hat{\omega}_\kappa^{(0)ab}\hat{e}_\mu^{(0)c}\partial_\nu\hat{\omega}_\lambda^{(0)de}\right) \\ &\times\ ((\eta_{ec}\eta_{af}-\eta_{fc}\eta_{ae})\gamma_b+(\eta_{fc}\eta_{be}-\eta_{ec}\eta_{bf})\gamma_a) \\ &+\ \hat{\omega}_\kappa^{(0)ab}\partial_\rho\hat{e}_\mu^{(0)c}\partial_\nu\hat{\omega}_\lambda^{(0)de}\epsilon_{dec}{}^d\epsilon_{abd}{}^m\gamma_m\Big]. \end{aligned} \quad (4.25)$$

To second order in $\theta^{\kappa\lambda}$, the resulting components along γ_a and $\gamma_5\gamma_a$ are

$$\begin{aligned} \hat{e}_\mu^{(2)h} &= -\frac{1}{4}\theta^{\kappa\lambda}\left(2\hat{a}_\kappa^{(1)}\partial_\lambda\hat{e}_\mu^{(0)h}-\frac{1}{2}\hat{\omega}_\kappa^{(0)ab}\partial_\lambda\hat{e}_{\mu5}^{(1)c}\epsilon_{abc}{}^h\right) \\ &-\frac{1}{4}\theta^{\kappa\lambda}\left(\hat{a}_\kappa^{(1)}\hat{e}_\mu^{(0)c}\hat{\omega}_\lambda^{(0)ah}\eta_{ac}-\frac{1}{4}\hat{\omega}_\kappa^{(0)ab}\hat{e}_{\mu5}^{(1)c}\hat{\omega}_\lambda^{(0)de}\eta_{dc}\epsilon_{abe}{}^h\right. \\ &\left.-\frac{1}{2}\hat{\omega}_\kappa^{(0)ab}\hat{e}_\mu^{(0)c}\hat{b}_{\lambda5}^{(1)}\epsilon_{abc}{}^h\right) \\ &+\frac{1}{32}\theta^{\kappa\lambda}\theta^{\sigma\nu}\times\Big[\left(\partial_\sigma\hat{\omega}_\kappa^{(0)ah}\partial_\nu\hat{e}_\mu^{(0)c}\hat{\omega}_\lambda^{(0)ef}+\partial_\sigma\hat{\omega}_\kappa^{(0)ah}\hat{e}_\mu^{(0)c}\partial_\nu\hat{\omega}_\lambda^{(0)ef}\right) \\ &\times(\eta_{ec}\eta_{af}-\eta_{fc}\eta_{ae})+\frac{1}{2}\hat{\omega}_\kappa^{(0)ab}\partial_\nu\hat{e}_\mu^{(0)c}\partial_\sigma\hat{\omega}_\lambda^{(0)ef}\epsilon_{efc}{}^d\epsilon_{abd}^h \\ &+4\partial_\sigma\hat{\omega}_\kappa^{(0)ah}\partial_\nu\partial_\lambda\hat{e}_\mu^{(0)c}\eta_{ac}\Big], \end{aligned} \quad (4.26)$$

$$\hat{e}_{\mu5}^{(2)a} = 0. \quad (4.27)$$

respectively.

As far as the spin-connection is concerned, one has (recall that $\frac{1}{2}\hat{R}_{\lambda\gamma}^{(1)} = \mathrm{i}r$, $\frac{\mathrm{i}}{2}\hat{R}_{\lambda\gamma5}^{(1)} = \tilde{r}$)

$$\omega_\gamma^2 = -\frac{1}{8}\theta^{\kappa\lambda}\left(\{\omega_\kappa^1,\partial_\lambda\omega_\gamma+R_{\lambda\gamma}\}+\{\omega_\kappa,\partial_\lambda\omega_\gamma^1+R_{\lambda\gamma}^1\}\right) \quad (4.28)$$

$$-\frac{\mathrm{i}}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}[\partial_\mu\omega_\kappa, \partial_\nu(\partial_\lambda\omega_\gamma + R_{\lambda\gamma})]$$

from which, by insertion of the various terms, we find (cf. Ref. [14])

$$\begin{aligned}\omega^2_\gamma \;=\; & -\frac{1}{8}\theta^{\kappa\lambda}\Big(2\hat{a}^{(1)}_\kappa(\frac{1}{2}\partial_\lambda\hat{\omega}^{(0)ab}_\gamma + \frac{\mathrm{i}}{2}\hat{R}^{(0)ab}_{\lambda\gamma})\sigma_{ab} - \frac{1}{2}\hat{b}^{(1)}_{\kappa 5}(\partial_\lambda\hat{\omega}^{(0)ab}_\gamma + \hat{R}^{(0)ab}_{\lambda\gamma})\epsilon_{abcd}\sigma^{cd} \\ & +\hat{\omega}^{(0)ab}_\kappa(\partial_\lambda\hat{a}^{(1)}_\gamma + \mathrm{i}\hat{r}^{(1)}_{\lambda\gamma})\sigma_{ab} + \frac{1}{2}\hat{\omega}^{(0)ab}_\kappa(\mathrm{i}\partial_\lambda\hat{b}^{(1)}_{\gamma 5} + \hat{\tilde{r}}^{(1)}_{\lambda\gamma})\epsilon_{abcd}\sigma^{cd}\Big) \\ & +\frac{1}{64}\theta^{\kappa\lambda}\theta^{\mu\nu}\partial_\mu\hat{\omega}^{(0)ab}_\kappa\partial_\nu(\partial_\lambda\hat{\omega}^{(0)cd}_\gamma + \frac{\mathrm{i}}{2}\hat{R}^{(0)cd}_{\lambda\gamma})(\eta_{ac}\sigma_{bd} - \eta_{bc}\sigma_{ad} - \eta_{ad}\sigma_{bc} + \eta_{bd}\sigma_{ac}), \end{aligned} \tag{4.29}$$

from which the components along σ_{ab}, I, γ_5 read as

$$\begin{aligned}\hat{\omega}^{(2)ab}_\gamma = & -\frac{1}{8}\theta^{\kappa\lambda}\Big(2\hat{a}^{(1)}_\kappa(\frac{1}{2}\partial_\lambda\hat{\omega}^{(0)ab}_\gamma + \hat{R}^{(0)ab}_{\lambda\gamma}) - \frac{1}{2}\hat{b}^{(1)}_{\kappa 5}(\partial_\lambda\hat{\omega}^{(0)cd}_\gamma + \mathrm{i}\hat{R}^{(0)cd}_{\lambda\gamma})\epsilon_{abcd} \\ & +\hat{\omega}^{(0)ab}_\kappa(\partial_\lambda\hat{a}^{(1)}_\gamma + \mathrm{i}\hat{r}^{(1)}_{\lambda\gamma}) + \frac{1}{2}\hat{\omega}^{(0)cd}_\kappa(\mathrm{i}\partial_\lambda\hat{b}^{(1)}_{\gamma 5} + \hat{\tilde{r}}^{(1)}_{\lambda\gamma})\epsilon_{abcd}\Big) \\ & +\frac{1}{64}\theta^{\kappa\lambda}\theta^{\mu\nu}\partial_\mu\hat{\omega}^{(0)ef}_\kappa\partial_\nu(\partial_\lambda\hat{\omega}^{(0)cd}_\gamma + \frac{\mathrm{i}}{2}\hat{R}^{(0)cd}_{\lambda\gamma})(\eta_{ec}\delta^a_f\delta^b_d - \eta_{fc}\delta^a_e\delta^b_d - \eta_{ed}\delta^a_f\delta^b_c + \eta_{fd}\delta^a_e\delta^b_c) \end{aligned} \tag{4.30}$$

$$\hat{a}^{(2)}_\gamma = \omega^{(2)}_\gamma = 0, \tag{4.31}$$

$$\hat{b}^{(2)}_{\gamma 5} = \tilde{\omega}^{(2)}_\gamma = 0. \tag{4.32}$$

Since the second-order part of the $\wedge_\star$ product of 1-forms is skew-symmetric, one has in our case

$$\begin{aligned} & R^{(2)ab} = d\omega^{(2)ab} + \omega^{(0)b}_c \wedge_{\star II} \omega^{(0)ca} \\ & +\omega^{(2)b}_c \wedge \omega^{(0)ca} + \omega^{(0)b}_c \wedge \omega^{(2)ca} + 2\mathrm{i}\omega^{(0)ab} \wedge_{\star I} \omega^{(1)} - \epsilon^{ab}{}_{cd}\omega^{(0)cd} \wedge_{\star I} \tilde{\omega}^{(1)}, \end{aligned} \tag{4.33}$$

$$\begin{aligned} & r^{(2)} = 0, \\ & \tilde{r}^{(2)} = 0. \end{aligned} \tag{4.34}$$

A neater geometric derivation of the second-order Seiberg–Witten map for pure gravity can be found in Ref. [15], with emphasis on the Seiberg–Witten differential equation for the action itself.

V. SECOND-ORDER ACTION FUNCTIONAL

The Aschieri–Castellani action for pure gravity [15–18] reads as

$$S = \int R^{ab} \wedge_\star (\hat{e}^c \wedge_\star \hat{e}^d - \hat{e}^c_5 \wedge_\star \hat{e}^d_5)\epsilon_{abcd}$$

$$
\begin{aligned}
&+2\mathrm{i}\ R^{ab} \wedge_\star (-\hat{e}_a \wedge_\star \hat{e}_{b5} + \hat{e}_{a5} \wedge_\star \hat{e}_b) \\
&+4\mathrm{i}\ r \wedge_\star (\hat{e}^a \wedge_\star \hat{e}_{a5} - \hat{e}_5^a \wedge_\star \hat{e}_a) \\
&+4\mathrm{i}\ \tilde{r} \wedge_\star (\hat{e}^a \wedge_\star \hat{e}_a - \hat{e}_5^a \wedge_\star \hat{e}_{a5}).
\end{aligned}
\tag{5.1}
$$

Bearing in mind that $r^{(0)} = \tilde{r}^{(0)} = R^{ab(1)} = \hat{e}^{(1)} = \hat{e}_5^{(0)c} = 0$, its form to second order in θ reduces to

$$
\begin{aligned}
S^{(2)} = & \int [R^{ab(0)} \wedge_\star \hat{e}^{(0)c} \wedge_\star \hat{e}^{(0)d} + R^{ab(0)} \wedge_\star (\hat{e}^{(0)c} \wedge \hat{e}^{(2)d} + \hat{e}^{(2)c} \wedge \hat{e}^{(0)d} - \hat{e}_5^{(1)c} \wedge \hat{e}_5^{(1)d}) \\
& +R^{ab(2)} \wedge_\star \hat{e}^{(0)c} \wedge_\star \hat{e}^{(0)d}]\epsilon_{abcd} \\
& +2\mathrm{i}\ R^{ab(0)} \wedge_\star (-\hat{e}_a^{(0)} \wedge_\star \hat{e}_{b5}^{(1)} + \hat{e}_{a5}^{(1)} \wedge_\star \hat{e}_b^{(0)}) \\
& +4\mathrm{i}\ r^{(1)} \wedge_\star (\hat{e}^{(0)a} \wedge_\star \hat{e}_{a5}^{(1)} - \hat{e}_5^{(1)a} \wedge_\star \hat{e}^{(0)a}) \\
& +4\mathrm{i}\ \tilde{r}^{(1)} \wedge_\star (\hat{e}^a \wedge_\star \hat{e}_a - \hat{e}_5^a \wedge_\star \hat{e}_{a5}).
\end{aligned}
\tag{5.2}
$$

Since the $\wedge_\star$ product of 1-forms has a first-order part in θ which is symmetric, while its second-order term is skew-symmetric, one finds

$$
\begin{aligned}
S^{(2)} = & \int \Big[R^{ab(0)} \wedge_{\star II} \hat{e}^{(0)c} \wedge \hat{e}^{(0)d} + R^{ab(0)} \wedge \hat{e}^{(0)c} \wedge_{\star II} \hat{e}^{(0)d} + R^{ab(0)} \wedge (\hat{e}^{(0)c} \wedge_{\star II} \hat{e}^{(0)d} \\
& + \hat{e}^{(0)c} \wedge \hat{e}^{(2)d} + \hat{e}^{(2)c} \wedge \hat{e}^{(0)d} - \hat{e}_5^{(1)c} \wedge \hat{e}_5^{(1)d}) \\
& + R^{ab(2)} \wedge \hat{e}^{(0)c} \wedge \hat{e}^{(0)d}]\epsilon_{abcd} - 4\mathrm{i}\ R^{ab(0)} \wedge \hat{e}_a^{(0)} \wedge_{\star I} \hat{e}_{b5}^{(1)} \\
& + 8\mathrm{i}\ r^{(1)} \wedge \hat{e}^{(0)a} \wedge \hat{e}_{a5}^{(1)} + 4\mathrm{i}\ \tilde{r}^{(1)} \wedge \hat{e}^a \wedge_{\star I} \hat{e}_a.
\end{aligned}
\tag{5.3}
$$

Variation of the action with respect to the components of the tetrad, i.e. $\hat{V}^a$ e $\widetilde{\hat{V}}^a$, yields the non-commutative field equations first obtained in Ref. [19]

$$
\begin{aligned}
& -\Big(\hat{V}^d \wedge_* \hat{R}^{ab} + \hat{R}^{ab} \wedge_* \hat{V}^d\Big)\varepsilon_{abcd} + \mathrm{i}(\eta_{bc}\eta_{ad} - \eta_{ac}\eta_{bd})\Big(\hat{R}^{ab} \wedge_* \widetilde{\hat{V}}^d - \widetilde{\hat{V}}^d \wedge_* \hat{R}^{ab}\Big) \\
& + 4\mathrm{i}\eta_{dc}\Big(\widetilde{\hat{r}} \wedge_* \hat{V}^d - \hat{V}^d \wedge_* \widetilde{\hat{r}}\Big) + 4\eta_{dc}\Big(\widetilde{\hat{V}}^d \wedge_* \hat{r} + \hat{r} \wedge_* \widetilde{\hat{V}}^d\Big) = 0,
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
& -\Big(\widetilde{\hat{V}}^d \wedge_* \hat{R}^{ab} + \hat{R}^{ab} \wedge_* \widetilde{\hat{V}}^d\Big)\varepsilon_{abcd} + \mathrm{i}(\eta_{bc}\eta_{ad} - \eta_{ac}\eta_{bd})\Big(\hat{R}^{ab} \wedge_* \hat{V}^d - \hat{V}^d \wedge_* \hat{R}^{ab}\Big) \\
& + 4\mathrm{i}\eta_{dc}\Big(\widetilde{\hat{r}} \wedge_* \widetilde{\hat{V}}^d - \widetilde{\hat{V}}^d \wedge_* \widetilde{\hat{r}}\Big) + 4\eta_{dc}\Big(\hat{V}^d \wedge_* \hat{r} + \hat{r} \wedge_* \hat{V}^d\Big) = 0.
\end{aligned}
\tag{5.5}
$$

To second order in θ, the field equation for $\widetilde{\hat{V}}^a$ is identically satisfied, while the one for $\hat{V}^a$, given in (5.4), reads as

$$2\Big[-\Big(\hat{V}^{d0}\wedge_{*II}\hat{R}^{ab0}+\hat{V}^{d2}\wedge\hat{R}^{ab0}+\hat{V}^{d0}\wedge\hat{R}^{ab2}\Big)\varepsilon_{abcd}+\mathrm{i}(\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd})\hat{R}^{ab0}\wedge_{*I}\widetilde{\hat{V}}^d$$
$$+4\mathrm{i}\eta_{dc}\widetilde{\hat{r}}\wedge_{*I}\hat{V}^d+4\eta_{dc}\widetilde{\hat{V}}^d\wedge\hat{r}\Big]=0, \tag{5.6}$$

VI. CLASSICAL TETRADS WHICH SOLVE THE VACUUM EINSTEIN EQUATIONS

Since in Ref. [19] some of us had found that only half of the non-commutative field equations are satisfied to first order in θ, when the classical tetrad describes the Schwarzschild solution in the standard coordinates, we have tried to check the result both in other coordinates for Schwarzschild, and for other solutions of the vacuum Einstein equations.

A. Classical tetrad in Kruskal–Szekeres coordinates

The Kruskal–Szekeres coordinates are very convenients because they make it evident that $r=0$ is only a coordinate singularity of the Schwarzschild geometry. Following [20], we write $(\sigma,\tau,\theta,\phi)$ for the Kruskal–Szekeres coordinates, with classical tetrad having components

$$\begin{aligned}
V^{(\sigma)} &= \frac{\tau\sqrt{A(r(\sigma,\tau))}}{\sqrt{2}\,\alpha}d\sigma+\frac{\alpha\sqrt{A(r(\sigma,\tau))}}{\sqrt{2}\,\tau}d\tau,\\
V^{(\tau)} &= \frac{\tau\sqrt{A(r(\sigma,\tau))}}{\sqrt{2}\,\alpha}d\sigma-\frac{\alpha\sqrt{A(r(\sigma,\tau))}}{\sqrt{2}\,\tau}d\tau,\\
V^{(\theta)} &= r(\sigma,\tau)d\theta,\\
V^{(\tau)} &= r(\sigma,\tau)\sin\theta d\phi,
\end{aligned} \tag{6.1}$$

where $A(r(\sigma,\tau))=\frac{4\lambda^3}{r}\exp^{-r/\lambda}=\frac{-2\lambda^3}{r}(\frac{r}{\lambda}-1)\frac{1}{\sigma\tau}$, and r is defined implicitly by $-\sigma\tau=(\frac{r}{\lambda}-1)\exp^{r/\lambda}$. Hence we find

$$\begin{aligned}
r^{(1,0)} &= \frac{-2\,\lambda^2\,\tau}{\mathrm{e}^{\frac{r(\sigma,\tau)}{\lambda}}\,r(\sigma,\tau)},\\
r^{(0,1)} &= \frac{-2\,\lambda^2\,\sigma}{\mathrm{e}^{\frac{r(\sigma,\tau)}{\lambda}}\,r(\sigma,\tau)},\\
r^{(1,1)} &= \frac{-2\,\lambda^2}{\mathrm{e}^{\frac{r(\sigma,\tau)}{\lambda}}\,(\lambda+r(\sigma,\tau))},
\end{aligned} \tag{6.2}$$

and the resulting spin-connection reads as [20]

$$
\begin{aligned}
\omega_1^{01} &= -\frac{1}{2}\left(1-\frac{\lambda^2}{r^2}\right),\\
\omega_2^{01} &= -\frac{1}{2}\left(1+\frac{\lambda^2}{r^2}\right),\\
\omega_3^{23} &= -\cos\theta,\\
\omega_2^{02} &= -\sqrt{\frac{A}{2}}\frac{1}{2\lambda}\left(\frac{\sigma\tau}{\alpha}+\alpha\right),\\
\omega_3^{03} &= -\sqrt{\frac{A}{2}}\frac{\sin\theta}{2\lambda}\left(\frac{\sigma\tau}{\alpha}+\alpha\right),\\
\omega_2^{12} &= -\sqrt{\frac{A}{2}}\frac{1}{2\lambda}\left(\frac{\sigma\tau}{\alpha}-\alpha\right),\\
\omega_3^{13} &= -\sqrt{\frac{A}{2}}\frac{\sin\theta}{2\lambda}\left(\frac{\sigma\tau}{\alpha}-\alpha\right),
\end{aligned}
\tag{6.3}
$$

where the labels $(1,2,3,4)$ refer to the coordinates $(\sigma,\tau,\theta,\phi)$, respectively.

If the only non-vanishing component of non-commutativity is θ^{23}, the non-commutative field equations reduce, to first order in θ^{23}, to the form [19]

$$
K_{c123}dx^1\wedge dx^2\wedge dx^3+K_{c124}dx^1\wedge dx^2\wedge dx^4+K_{c134}dx^1\wedge dx^3\wedge dx^4+K_{c234}dx^2\wedge dx^3\wedge dx^4. \tag{6.4}
$$

For the first-order field equations to hold, this should vanish identically, which is not the case because, on the contrary,

$$
\begin{aligned}
K_{c123} = \quad & \delta_{c1}\frac{1}{\mathrm{e}^{\frac{3\,r(\sigma,\tau)}{\lambda}}\,\alpha\,\tau\,r(\sigma,\tau)^5\,(\lambda+r(\sigma,\tau))^2}\\
\times \quad & 4\sqrt{2}\,\lambda\sqrt{\frac{\lambda^3}{\mathrm{e}^{\frac{r(\sigma,\tau)}{\lambda}}\,r(\sigma,\tau)}}\,\Big(\mathrm{e}^{\frac{2\,r(\sigma,\tau)}{\lambda}}\,r(\sigma,\tau)^5\,\left(3\,\lambda\,\left(\alpha^2+\sigma\,\tau\right)+\left(2\,\alpha^2+3\,\sigma\,\tau\right)\,r(\sigma,\tau)\right)\\
& +\mathrm{e}^{\frac{r(\sigma,\tau)}{\lambda}}\,\lambda\,\sigma\,\tau\,r(\sigma,\tau)^2\,(\lambda+r(\sigma,\tau))^2\,\left(-8\,\alpha^2\,\lambda+4\,\lambda\,\sigma\,\tau+\left(-3\,\alpha^2+5\,\sigma\,\tau\right)\,r(\sigma,\tau)\right)\\
& +4\,\lambda^2\,\sigma^2\,\tau^2\,\Big(\lambda+r(\sigma,\tau)\Big)^2\,\Big(\lambda^2\,\Big(-5\,\alpha^2+3\,\sigma\,\tau\Big)\\
- \quad & 2\,\left(2\,\alpha^2-\sigma\,\tau\right)\,r(\sigma,\tau)\,(2\,\lambda+r(\sigma,\tau))\Big)\Big),\\
K_{c124} = \quad & \delta_{c4}\mathrm{e}^{-\frac{3\,r(\sigma,\tau)}{\lambda}}8\,\lambda\,\cos(\theta)\\
& \left(\frac{2\,\lambda^4\,\sigma^2\,\tau\,(\lambda+r(\sigma,\tau))}{r(\sigma,\tau)^5}+\frac{\mathrm{e}^{\frac{2\,r(\sigma,\tau)}{\lambda}}\,(3\,\lambda+r(\sigma,\tau))}{\tau\,(\lambda+r(\sigma,\tau))}\right.\\
& \left.+\,\frac{\mathrm{e}^{\frac{r(\sigma,\tau)}{\lambda}}\,\lambda\,\sigma\,\left(2\,\lambda^2+r(\sigma,\tau)\,(\lambda+r(\sigma,\tau))\right)}{r(\sigma,\tau)^3}\right),\\
K_{c134} = \quad & \delta_{c4}\frac{\lambda}{\mathrm{e}^{\frac{3\,r(\sigma,\tau)}{\lambda}}\,r(\sigma,\tau)^4\,(\lambda+r(\sigma,\tau))^2}
\end{aligned}
$$

$$4\left(-\left(\mathrm{e}^{\frac{2\,r(\sigma,\tau)}{\lambda}}\,r(\sigma,\tau)^6\right)-2\,\mathrm{e}^{\frac{r(\sigma,\tau)}{\lambda}}\,\lambda\,\sigma\,\tau\,r(\sigma,\tau)^2\,(\lambda+r(\sigma,\tau))\,(2\,\lambda+r(\sigma,\tau))^2\right.$$
$$\left.-\,4\,\lambda^2\,\sigma^2\,\tau^2\,(\lambda+r(\sigma,\tau))^2\,\left(5\,\lambda^2+4\,r(\sigma,\tau)\,(2\,\lambda+r(\sigma,\tau))\right)\right)\,\sin(\theta),$$

$$K_{c234} = \quad \delta_{c4}\frac{1}{\mathrm{e}^{\frac{3\,r(\sigma,\tau)}{\lambda}}\,r(\sigma,\tau)^4\,(\lambda+r(\sigma,\tau))}$$
$$8\,\lambda^2\,\sigma^2\,\left(-\left(\mathrm{e}^{\frac{r(\sigma,\tau)}{\lambda}}\,r(\sigma,\tau)^2\,\left(2\,\lambda^2+3\,\lambda\,r(\sigma,\tau)+2\,r(\sigma,\tau)^2\right)\right)\right.$$
$$\left.-\,2\,\lambda\,\sigma\,\tau\,(\lambda+r(\sigma,\tau))\,\left(3\,\lambda^2+2\,r(\sigma,\tau)\,(2\,\lambda+r(\sigma,\tau))\right)\right)\,\sin(\theta). \tag{6.5}$$

B. The Plebanski case

We here consider the class of vacuum space-times with two commuting Killing vector fields $\frac{\partial}{\partial\sigma}$ and $\frac{\partial}{\partial\tau}$ first found by Plebanski, whose tetrad [21] reads as

$$\begin{aligned}
V^{(\sigma)} &= -\sqrt{C(q,p)}\mathrm{d}\sigma,\\
V^{(q)} &= \frac{(-p+q)^{\frac{1-(\beta-\gamma)^2}{2}}\,(p+q)^{\frac{1-(\beta+\gamma)^2}{2}}\,\sqrt{A(q,p)}}{\sqrt{-1+q^2}}\mathrm{d}q,\\
V^{(p)} &= \frac{(-p+q)^{\frac{1-(\beta-\gamma)^2}{2}}\,(p+q)^{\frac{1-(\beta+\gamma)^2}{2}}\,\sqrt{A(q,p)}}{\sqrt{1-p^2}}\mathrm{d}p,\\
V^{(\tau)} &= -\sqrt{B(q,p)}\mathrm{d}\tau,
\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}
A(q,p) &\equiv m^2(q+1)^{\alpha^2-1/4}(q-1)^{\beta^2-1/4}(p+1)^{\gamma^2-1/4}(1-p)^{\delta^2-1/4},\\
B(q,p) &\equiv m^2(q+1)^{\alpha+1/2}(q-1)^{\beta+1/2}(p+1)^{\gamma+1/2}(1-p)^{\delta+1/2}\\
C(q,p) &\equiv m^2(q+1)^{1/2-\alpha}(q-1)^{1/2-\beta}(p+1)^{1/2-\gamma}(1-p)^{1/2-\delta}\\
\alpha &= a+b,\ \beta=a-b,\ \gamma\ = a+c,\ \delta=a-c,
\end{aligned} \tag{6.7}$$

where a,b,c,m are real constants, and the coordinates (p,q) lie in the intervals $-1<p<1$, $q<\infty$, respectively.

The components of non-commutative field equations take the form

$$\begin{aligned}
K_{c123} &= \delta_{c1}F(p,q),\\
K_{c124} &= 0,\\
K_{c134} &= 0,\\
K_{c234} &= \delta_{c4}G(p,q),
\end{aligned} \tag{6.8}$$

where

$$
\begin{aligned}
F(p,q) &\equiv \frac{1}{2\,\sqrt{-1+p^2}\,(p-q)^2\,(p+q)^2\,\sqrt{-1+q^2}\,A(p,q)^2} \\
\times &\left(\frac{-\left(\sqrt{-1+q^2}\,\left(2\,\left(-2\,\beta\,\gamma\,p+\beta^2\,q+\left(-1+\gamma^2\right)\,q\right)\,A(p,q)+\left(p^2-q^2\right)\,A^{(0,1)}(p,q)\right)\,C^{(0,1)}(p,q)\right)}{4\,\sqrt{-1+p^2}\,(p-q)\,(p+q)\,A(p,q)\,\sqrt{C(p,q)}}\right. \\
&\left.+\frac{\sqrt{-1+p^2}\,\left(-2\,\left(\left(-1+\beta^2+\gamma^2\right)\,p-2\,\beta\,\gamma\,q\right)\,A(p,q)+\left(p^2-q^2\right)\,A^{(1,0)}(p,q)\right)\,C^{(1,0)}(p,q)}{4\,(-p+q)\,(p+q)\,\sqrt{-1+q^2}\,A(p,q)\,\sqrt{C(p,q)}}\right) \\
\times &\left(4\,\left(2\,\beta\,\gamma\,p^3\,q+\left(-1+\beta^2+\gamma^2\right)\,q^2+2\,\beta\,\gamma\,p\,q\,\left(-2+q^2\right)\right.\right. \\
&\left.-\left(-1+\beta^2+\gamma^2\right)\,p^2\,\left(-1+2\,q^2\right)\right)\,A(p,q)^2 \\
&+\left(p^2-q^2\right)^2\,\left(\left(-1+q^2\right)\,A^{(0,1)}(p,q)^2+\left(-1+p^2\right)\,A^{(1,0)}(p,q)^2\right) \\
&-\left(p^2-q^2\right)^2\,A(p,q)\,\left(q\,A^{(0,1)}(p,q)+\left(-1+q^2\right)\,A^{(0,2)}(p,q)+p\,A^{(1,0)}(p,q)\right. \\
&\left.\left.-\,A^{(2,0)}(p,q)+p^2\,A^{(2,0)}(p,q)\right)\right)\,, \qquad (6.9)
\end{aligned}
$$

while G is given by too lengthy an expression for a regular paper.

C. Vacuum solutions of class AIII

The degenerate static vacuum solutions of class AIII [21] can be described by using the coordinates (t,r,z,ϕ) to which no Killing vectors can be associated. The corresponding classical tetrad has components [21]

$$
\begin{aligned}
V^{(t)} &= -\frac{1}{\sqrt{z}}\mathrm{d}t, \\
V^{(z)} &= \sqrt{z}\,\mathrm{d}z, \\
V^{(r)} &= z\,\mathrm{d}r, \\
V^{(\phi)} &= rz\mathrm{d}\phi. \qquad (6.10)
\end{aligned}
$$

To first order in non-commutativity, the components of field equations turn out to be

$$
\begin{aligned}
K_{c123} &= \delta_{c1}\frac{-5}{z^{\frac{7}{2}}}, \\
K_{c124} &= 0, \\
K_{c134} &= 0, \\
K_{c234} &= \delta_{c4}\frac{-2\,r}{z^2}. \qquad (6.11)
\end{aligned}
$$

At this stage, one might think that the problem lies in the attempt of finding solutions of the non-commutative field equations via the Seiberg–Witten map. For this purpose, we show in the appendix what happens if the Seiberg–Witten map is not used for finding solutions of non-commutative field equations.

VII. THE SIMPLER CASE OF VACUUM MAXWELL THEORY

Since in [19] it had been found that inserting the Seiberg–Witten map in the non-commutative field equations is inequivalent to inserting it directly in the action for pure gravity, we here consider the simpler case of vacuum Maxwell theory, for which the non-commutative potential, gauge parameter and field strength read as

$$\widehat{A}_\mu = A_\mu + \widetilde{A}_\mu + \mathrm{O}(\theta^2), \tag{7.1}$$

$$\widehat{\Lambda} = \Lambda + \widetilde{\Lambda} + \mathrm{O}(\theta^2), \tag{7.2}$$

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - \mathrm{i}[\widehat{A}_\mu, \widehat{A}_\nu]_\star = F^{(0)}_{\mu\nu} + \widetilde{F}_{\mu\nu} + \mathrm{O}(\theta^2), \tag{7.3}$$

where

$$\widetilde{F}_{\mu\nu} = \partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu + \theta^{\rho\sigma}\partial_\rho A_\mu \partial_\sigma A_\nu. \tag{7.4}$$

The equation defining the Seiberg–Witten map

$$\widehat{\delta}_{\widehat{\Lambda}} \widehat{A}_\mu(A) = \widehat{A}_\mu(A + \delta_\Lambda A) - \widehat{A}_\mu(A) \tag{7.5}$$

is solved, to first order in $\theta^{\rho\sigma}$, by

$$\widehat{A}_\mu = -\frac{\theta^{\rho\sigma}}{2}\Big(A_\rho \partial_\sigma A_\mu + A_\rho F_{\sigma\mu}\Big), \tag{7.6}$$

$$\widehat{\Lambda} = \frac{\theta^{\rho\sigma}}{2}(\partial_\rho \Lambda) A_\sigma. \tag{7.7}$$

Variation of the action functional to first order in θ yields now equations in agreement with the insertion of the Seiberg–Witten map in the non-commutative field equations

$$\widehat{D}^\mu \widehat{F}_{\mu\nu} = 0. \tag{7.8}$$

In both cases, we find eventually

$$\theta^{\rho\sigma} u_{\rho\sigma\nu} = 0, \tag{7.9}$$

where

$$
\begin{aligned}
u_{\rho\sigma\nu} &= -(\Box A_\rho)(\partial_\sigma A_\nu) + \frac{1}{2}\Big[(\Box A_\rho)(\partial_\nu A_\sigma) - (\Box A_\sigma)(\partial_\nu A_\rho)\Big] \\
&+ \frac{1}{2} A_\rho \Big[\partial_\nu(\Box A_\sigma) - \partial_\sigma(\Box A_\nu)\Big] \\
&+ (\partial_\mu A_\rho)\Big[\frac{1}{2}\partial^\mu \partial_\nu A_\sigma - \partial^\mu \partial_\sigma A_\nu\Big] \\
&+ (\partial^\mu A_\rho)\Big[\partial^2_{\nu\sigma} A_\mu - \partial^2_{\mu\sigma} A_\nu\Big] \\
&+ \frac{1}{2} A_\rho \Big[2\partial^\mu \partial^2_{\nu\sigma} A_\mu + \partial^\mu(\partial^2_{\mu\sigma} A_\nu - \partial^2_{\mu\nu} A_\sigma)\Big] \\
&+ (\partial^\mu \partial_\nu A_\rho)\Big[\partial_\sigma A_\mu - \frac{1}{2}\partial_\mu A_\sigma\Big] \\
&+ \frac{1}{2}(\partial_\nu A_\rho)\Big[2\partial^\mu \partial_\sigma A_\mu - \Box A_\sigma\Big] \\
&+ (\partial_\rho A^\mu)(\partial^2_{\mu\sigma} A_\nu - \partial^2_{\nu\sigma} A_\mu) \\
&+ \Big[(\partial_\rho A_\mu)(\partial^\mu \partial_\sigma A_\nu) + (\partial_\sigma A_\nu)(\partial^\mu \partial_\rho A_\mu)\Big].
\end{aligned}
\tag{7.10}
$$

This is a sort of *off-shell formula*, because we have not assumed that the classical background solves the vacuum Maxwell equations. For example, in the Lorenz gauge $\partial^\nu A_\nu = 0$, A_μ obeys the wave equation $\Box A_\mu = 0$, and $u_{\rho\sigma\nu}$ takes the *on-shell form*

$$
\begin{aligned}
\tilde{u}_{\rho\sigma\nu} &= \frac{1}{2}(\partial_\mu A_\rho)\Big(\partial^\mu \partial_\nu A_\sigma - 2\partial^\mu \partial_\sigma A_\nu\Big) \\
&+ (\partial^\mu A_\rho)(\partial^2_{\nu\sigma} A_\mu - \partial^2_{\mu\sigma} A_\nu) \\
&+ \frac{1}{2} A_\rho \partial^\mu (\partial^2_{\mu\sigma} A_\nu - \partial^2_{\mu\nu} A_\sigma) \\
&+ \frac{1}{2}(\partial^\mu \partial_\nu A_\rho)\Big(2\partial_\sigma A_\mu - \partial_\mu A_\sigma\Big) \\
&+ (\partial_\rho A^\mu)\Big(\partial^2_{\mu\sigma} A_\nu - \partial^2_{\nu\sigma} A_\mu\Big) \\
&+ (\partial_\rho A_\mu)\Big(\partial^\mu \partial_\sigma A_\nu\Big).
\end{aligned}
\tag{7.11}
$$

Eventually, if $\theta^{\rho\sigma}$ reads as [19]

$$
\theta^{\rho\sigma} = \theta\Big(\delta^{\rho 2}\delta^{\sigma 3} - \delta^{\rho 3}\delta^{\sigma 2}\Big), \tag{7.12}
$$

we have to check whether $\theta^{\rho\sigma}\tilde{u}_{\rho\sigma\nu}$ vanishes when $\Box A_\mu = 0$. In Cartesian coordinates, on considering the simple plane-wave solution to Maxwell's equations in Lorenz gauge, i.e.

$$
A_\mu(t, x, y, z) = \mathrm{e}^{\mathrm{i}(\omega t - k\cdot x)} U_\mu, \tag{7.13}
$$

the components of $u_\nu = \theta^{\rho\sigma}\tilde{u}_{\rho\sigma\nu}$ turn out to be

$$\begin{aligned}
u_1 = \quad & -\frac{\mathrm{i}}{2}\,\mathrm{e}^{2\,\mathrm{i}(\omega t-\,(x\,\kappa_1+y\,\kappa_2+z\,\kappa_3))}\,\left(4\,\kappa_1\,\left(\omega\,U_4+U_2\,\kappa_2+U_3\,\kappa_3\right)+U_1\,\left(3\,\omega^2+{\kappa_1}^2-3\,{\kappa_2}^2-3\,{\kappa_3}^2\right)\right)\\
& \times\left(-\left(U_3\,\kappa_1\,\theta_{13}\right)+U_4\,\kappa_1\,\theta_{14}+U_1\,\left(\kappa_2\,\theta_{12}+\kappa_3\,\theta_{13}+\omega\,\theta_{14}\right)\right.\\
& \left.-\,U_3\,\kappa_2\,\theta_{23}+U_4\,\kappa_2\,\theta_{24}+U_2\,\left(-\left(\kappa_1\,\theta_{12}\right)+\kappa_3\,\theta_{23}+\omega\,\theta_{24}\right)+\omega\,U_3\,\theta_{34}+U_4\,\kappa_3\,\theta_{34}\right),\\
u_2 = \quad & -\frac{\mathrm{i}}{2}\,\mathrm{e}^{2\,\mathrm{i}(\omega t-\,(x\,\kappa_1+y\,\kappa_2+z\,\kappa_3))}\,\left(4\,\kappa_2\,\left(\omega\,U_4+U_1\,\kappa_1+U_3\,\kappa_3\right)+U_2\,\left(3\,\omega^2-3\,{\kappa_1}^2+{\kappa_2}^2-3\,{\kappa_3}^2\right)\right)\\
& \times\left(-\left(U_3\,\kappa_1\,\theta_{13}\right)+U_4\,\kappa_1\,\theta_{14}+U_1\,\left(\kappa_2\,\theta_{12}+\kappa_3\,\theta_{13}+\omega\,\theta_{14}\right)\right.\\
& \left.-\,U_3\,\kappa_2\,\theta_{23}+U_4\,\kappa_2\,\theta_{24}+U_2\,\left(-\left(\kappa_1\,\theta_{12}\right)+\kappa_3\,\theta_{23}+\omega\,\theta_{24}\right)+\omega\,U_3\,\theta_{34}+U_4\,\kappa_3\,\theta_{34}\right),\\
u_3 = \quad & -\frac{\mathrm{i}}{2}\,\mathrm{e}^{2\,\mathrm{i}(\omega t-\,(x\,\kappa_1+y\,\kappa_2+z\,\kappa_3))}\,\left(4\,\left(\omega\,U_4+U_1\,\kappa_1+U_2\,\kappa_2\right)\,\kappa_3+U_3\,\left(3\,\omega^2-3\,{\kappa_1}^2-3\,{\kappa_2}^2+{\kappa_3}^2\right)\right)\ ,\\
& \times\left(-\left(U_3\,\kappa_1\,\theta_{13}\right)+U_4\,\kappa_1\,\theta_{14}+U_1\,\left(\kappa_2\,\theta_{12}+\kappa_3\,\theta_{13}+\omega\,\theta_{14}\right)\right.\\
& \left.-U_3\,\kappa_2\,\theta_{23}+U_4\,\kappa_2\,\theta_{24}+U_2\,\left(-\left(\kappa_1\,\theta_{12}\right)+\kappa_3\,\theta_{23}+\omega\,\theta_{24}\right)+\omega\,U_3\,\theta_{34}+U_4\,\kappa_3\,\theta_{34}\right),\\
u_4 = \quad & -\frac{\mathrm{i}}{2}\,\mathrm{e}^{2\,\mathrm{i}(\omega t-\,(x\,\kappa_1+y\,\kappa_2+z\,\kappa_3))}\,\left(4\,\omega\,\left(U_1\,\kappa_1+U_2\,\kappa_2+U_3\,\kappa_3\right)+U_4\,\left(\omega^2+3\,{\kappa_1}^2+3\,{\kappa_2}^2+3\,{\kappa_3}^2\right)\right)\\
& \left(-\left(U_3\,\kappa_1\,\theta_{13}\right)+U_4\,\kappa_1\,\theta_{14}+U_1\,\left(\kappa_2\,\theta_{12}+\kappa_3\,\theta_{13}+\omega\,\theta_{14}\right)\right.\\
& \left.-U_3\,\kappa_2\,\theta_{23}+U_4\,\kappa_2\,\theta_{24}+U_2\,\left(-\left(\kappa_1\,\theta_{12}\right)+\kappa_3\,\theta_{23}+\omega\,\theta_{24}\right)+\omega\,U_3\,\theta_{34}+U_4\,\kappa_3\,\theta_{34}\right).
\end{aligned}\tag{7.14}$$

In the case $\theta_{23} \neq 0$, we find

$$\begin{aligned}
u_1 = \quad & \frac{\mathrm{i}}{2}\,\mathrm{e}^{2\,\mathrm{i}(\omega t-\,(x\,\kappa_1+y\,\kappa_2+z\,\kappa_3))}\,\theta_{23}\,\left(U_3\,\kappa_2-U_2\,\kappa_3\right)\,\left(4\,\kappa_1\,\left(\omega\,U_4+U_2\,\kappa_2+U_3\,\kappa_3\right)\right.\\
& \left.+U_1\,\left(3\,\omega^2+{\kappa_1}^2-3\,{\kappa_2}^2-3\,{\kappa_3}^2\right)\right),\\
u_2 = \quad & \frac{\mathrm{i}}{2}\,\mathrm{e}^{2\,\mathrm{i}(\omega t-\,(x\,\kappa_1+y\,\kappa_2+z\,\kappa_3))}\,\theta_{23}\,\left(-\left(U_3\,\kappa_2\right)+U_2\,\kappa_3\right)\,\left(4\,\kappa_2\,\left(\omega\,U_4+U_1\,\kappa_1+U_3\,\kappa_3\right)\right.\\
& \left.+U_2\,\left(3\,\omega^2-3\,{\kappa_1}^2+{\kappa_2}^2-3\,{\kappa_3}^2\right)\right),\\
u_3 = \quad & \frac{\mathrm{i}}{2}\,\mathrm{e}^{2\,\mathrm{i}(\omega t-\,(x\,\kappa_1+y\,\kappa_2+z\,\kappa_3))}\,\theta_{23}\,\left(U_3\,\kappa_2-U_2\,\kappa_3\right)\,\left(4\,\left(\omega\,U_4+U_1\,\kappa_1+U_2\,\kappa_2\right)\,\kappa_3\right.\\
& \left.+U_3\,\left(3\,\omega^2-3\,{\kappa_1}^2-3\,{\kappa_2}^2+{\kappa_3}^2\right)\right),\\
u_4 = \quad & \frac{\mathrm{i}}{2}\,\mathrm{e}^{2\,\mathrm{i}(\omega t-\,(x\,\kappa_1+y\,\kappa_2+z\,\kappa_3))}\,\theta_{23}\,\left(U_3\,\kappa_2-U_2\,\kappa_3\right)\,\left(4\,\omega\,\left(U_1\,\kappa_1+U_2\,\kappa_2+U_3\,\kappa_3\right)\right.\\
& \left.+U_4\,\left(\omega^2+3\,{\kappa_1}^2+3\,{\kappa_2}^2+3\,{\kappa_3}^2\right)\right),
\end{aligned}\tag{7.15}$$

and hence $u_\nu = \theta^{\rho\sigma}\tilde{u}_{\rho\sigma\nu} = 0$ is satisfied when

$$\frac{U_3}{U_2} = \frac{\kappa_3}{\kappa_2}.\tag{7.16}$$

For all $\theta_{ij} \neq 0$, $u_\nu = \theta^{\rho\sigma}\tilde{u}_{\rho\sigma\nu} = 0$ (see Eq. (7.14)) is satisfied when

$$\begin{aligned}
& U_i = \kappa_i,\ \ i = 1, 2, 3,\\
& U_4 = -\omega.
\end{aligned}\tag{7.17}$$

However, this is too particular a solution to first order, and its physical meaning, if any, is unclear to us.

In polar coordinates, on considering the simple spherical-wave solution to Maxwell's equations in Lorenz gauge, i.e.

$$A_{\mu}(t,r) = \frac{U_{\mu}}{r} \mathrm{e}^{\mathrm{i}(\omega t - |\kappa| r)}, \tag{7.18}$$

in the case $\theta_{23} \neq 0$, $u_{\nu} = \theta^{\rho\sigma} \tilde{u}_{\rho\sigma\nu} = 0$ is satisfied.

VIII. CONCLUDING REMARKS

For pure gravity, the non-commutative field equations first found in Ref. [19] cannot be both solved by inserting the Seiberg–Witten map, nor do we succeed in finding solutions which do not exploit such a map, as is shown in detail in the appendix. This is now tested for three backgrounds solving the vacuum Einstein equations. The second-order Seiberg–Witten map for pure gravity has also been evaluated explicitly in our paper (but we acknowledge the neater geometric results first obtained in Ref. [15]).

If one considers instead the non-commutative version of vacuum Maxwell theory, the stage at which the Seiberg–Witten map is inserted does not make any difference, but no obvious solution of the non-commutative field equations has been obtained so far by us on this side. Moreover, the gravitational counterpart of the off-shell versus on-shell analysis of Sec. VIII is lacking at present, and is also a topic for further investigation in our opinion.

As far as we can see, while much progress has been made in the literature on the Seiberg–Witten map to various orders and its geometric structure [15], it remains unclear whether the non-commutative field equations admit solutions. So far, the Seiberg–Witten map has been exploited to establish a correspondence between non-commutative and commutative fields, hence showing that once the non-commutative action is expanded in terms of the commutative fields, the resulting action is gauge-invariant under ordinary local Lorentz transformations [22, 23]. But our detailed calculations show that the problem of solving the non-commutative field equations remains open.

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Appendix A: Studying the non-commutative field equations without the Seiberg–Witten map

Let us start from the non-commutative field equation written in the form [19]

$$\Big[-\varepsilon_{abcd}\widetilde{V}^d_\mu R^{(0)ab}_{\nu\lambda} + \theta^{\rho\sigma}\Big(\partial_\rho V^d_\mu\Big)\Big(\partial_\sigma R^{(0)}_{dc\,\nu\lambda}\Big) + 4V_{c\mu}r^{(1)}_{\nu\lambda}\Big]dx^\mu \wedge dx^\nu \wedge dx^\lambda + \mathrm{O}(\theta^2) = 0, \quad \text{(A1)}$$

where we consider $\widetilde{V}^d_\mu$ as represented by a generic matrix

$$\begin{pmatrix} F^1_1(t,r,\theta,\phi) & F^1_2(t,r,\theta,\phi) & F^1_3(t,r,\theta,\phi) & F^1_4(t,r,\theta,\phi) \\ F^2_1(t,r,\theta,\phi) & F^2_2(t,r,\theta,\phi) & F^2_3(t,r,\theta,\phi) & F^2_4(t,r,\theta,\phi) \\ F^3_1(t,r,\theta,\phi) & F^3_2(t,r,\theta,\phi) & F^3_3(t,r,\theta,\phi) & F^3_4(t,r,\theta,\phi)\} \\ F^4_1(t,r,\theta,\phi) & F^4_2(t,r,\theta,\phi) & F^4_3(t,r,\theta,\phi) & F^4_4(t,r,\theta,\phi) \end{pmatrix}, \quad \text{(A2)}$$

and we write the components of ω_μ as $(\omega_1, \omega_2, \omega_3, \omega_4)$. In a Schwarzschild background, when we require fulfillment of the torsion constraint

$$T^a_{(cl)} \wedge_* V^b \eta_{ab}\gamma_5 + \Big(T^a_{(cl)} \wedge \widetilde{V}^b - \widetilde{T}^a \wedge V^b\Big)\eta_{ab} + \frac{\mathrm{i}}{2}T^c_{(I)} \wedge V^d \varepsilon_{cdab}\gamma^{ab} = 0, \quad \text{(A3)}$$

then its component along γ^{ab}, i.e. $G_{ab}[a,b,\mu,\nu,\sigma] = \frac{\mathrm{i}}{2}T^c_{(\mu\nu I)} \wedge V^d_\sigma \varepsilon_{cdab}$, should vanish. On the other hand, we find (writing for simplicity $(F_{ij} = F^i_j)$)

$$G_{ab}[1,2,2,3,4] = 4\,r^2\,\sin(\theta)\,\omega_2(t,r,\theta,\phi),$$

$$G_{ab}[1,2,1,3,4] = \frac{-\left(M\,F_{34}(t,r,\theta,\phi) - M\,\sin(\theta)\,F_{43}(t,r,\theta,\phi) + 8\,r^3\,\sin(\theta)\,\omega_1(t,r,\theta,\phi)\right)}{2\,r},$$

$$G_{ab}[1,2,1,2,4] = \frac{-\left(M\,\sin(\theta)\,F_{42}(t,r,\theta,\phi)\right)}{2\,r},$$

$$G_{ab}[1,2,1,2,3] = \frac{M\,F_{32}(t,r,\theta,\phi)}{2\,r},$$

$$G_{ab}[1,3,2,3,4] = \frac{-F_{14}(t,r,\theta,\phi)}{2} + \frac{4\,\sqrt{1-\frac{2\,M}{r}}\,r^2\,\sin(\theta)\,\omega_3(t,r,\theta,\phi)}{-2\,M + r},$$

$$G_{ab}[1,3,1,2,4] = \frac{-\left(M\,F_{34}(t,r,\theta,\phi) + 8\,r^3\,\sin(\theta)\,\omega_1(t,r,\theta,\phi)\right)}{2\,\sqrt{1-\frac{2\,M}{r}}\,r^2},$$

$$G_{ab}[1,3,1,2,3] = \frac{F_{11}(t,r,\theta,\phi) + \frac{M\,F_{33}(t,r,\theta,\phi)}{\sqrt{1-\frac{2\,M}{r}}\,r^2}}{2},$$

$$G_{ab}[1,4,2,3,4] = \frac{-\left(r\,\cos(\theta)\,F_{12}(t,r,\theta,\phi)\right) + \sin(\theta)\,F_{13}(t,r,\theta,\phi) + \frac{8\,\sqrt{1-\frac{2\,M}{r}}\,r^2\,\omega_4(t,r,\theta,\phi)}{-2\,M+r}}{2},$$

$$G_{ab}[1,4,1,3,4] = \frac{r\,\cos(\theta)\,F_{11}(t,r,\theta,\phi)}{2},$$

$$G_{ab}[1,4,1,2,4] = \frac{-\left(\sin(\theta)\,F_{11}(t,r,\theta,\phi)\right) - \frac{M\,F_{44}(t,r,\theta,\phi)}{\sqrt{1-\frac{2\,M}{r}}\,r^2}}{2},$$

$$G_{ab}[1,4,1,2,3] = \frac{M\,F_{43}(t,r,\theta,\phi) - 8\,r^3\,\omega_1(t,r,\theta,\phi)}{2\,\sqrt{1-\frac{2\,M}{r}}\,r^2},$$

$$G_{ab}[2,1,2,3,4] = -4\,r^2\,\sin(\theta)\,\omega_2(t,r,\theta,\phi),$$

$$G_{ab}[2,1,1,3,4] = \frac{M\,F_{34}(t,r,\theta,\phi) - M\,\sin(\theta)\,F_{43}(t,r,\theta,\phi) + 8\,r^3\,\sin(\theta)\,\omega_1(t,r,\theta,\phi)}{2\,r},$$

$$G_{ab}[2,1,1,2,4] = \frac{M\,\sin(\theta)\,F_{42}(t,r,\theta,\phi)}{2\,r},$$

$$G_{ab}[2,1,1,2,3] = \frac{-\left(M\,F_{32}(t,r,\theta,\phi)\right)}{2\,r},$$

$$G_{ab}[2,3,2,3,4] = \frac{-\left(\sqrt{1-\frac{2\,M}{r}}\,r\,\sin(\theta)\,F_{42}(t,r,\theta,\phi)\right)}{2},$$

$$G_{ab}[2,3,1,3,4] = \frac{(2\,M-r)\,F_{14}(t,r,\theta,\phi) + \sqrt{1-\frac{2\,M}{r}}\,r^2\,\sin(\theta)\,\left(F_{41}(t,r,\theta,\phi) + 8\,\omega_3(t,r,\theta,\phi)\right)}{2\,r},$$

$$G_{ab}[2,3,1,2,4] = -4\,\sqrt{1-\frac{2\,M}{r}}\,r\,\sin(\theta)\,\omega_2(t,r,\theta,\phi),$$

$$G_{ab}[2,3,1,2,3] = \frac{\left(1-\frac{2\,M}{r}\right)\,F_{12}(t,r,\theta,\phi)}{2},$$

$$G_{ab}[2,4,2,3,4] = \frac{-\left(r\,\left(\cos(\theta)\,F_{22}(t,r,\theta,\phi) - \sqrt{1-\frac{2\,M}{r}}\,\sin(\theta)\,F_{32}(t,r,\theta,\phi)\right)\right)}{2},$$

$$G_{ab}[2,4,1,3,4] = \frac{1}{2\,r}\,(-2\,M+r)\,\sin(\theta)\,F_{13}(t,r,\theta,\phi)$$
$$+r^2\left(\cos(\theta)\,F_{21}(t,r,\theta,\phi) + \sqrt{1-\frac{2\,M}{r}}\,\left(-\left(\sin(\theta)\,F_{31}(t,r,\theta,\phi)\right) + 8\,\omega_4(t,r,\theta,\phi)\right)\right),$$

$$G_{ab}[2,4,1,2,4] = \frac{-\left(\left(1-\frac{2\,M}{r}\right)\,\sin(\theta)\,F_{12}(t,r,\theta,\phi)\right)}{2},$$

$$G_{ab}[2,4,1,2,3] = -4\,\sqrt{1-\frac{2\,M}{r}}\,r\,\omega_2(t,r,\theta,\phi),$$

$$G_{ab}[3,1,2,3,4] = \frac{F_{14}(t,r,\theta,\phi)}{2} - \frac{4\,\sqrt{1-\frac{2\,M}{r}}\,r^2\,\sin(\theta)\,\omega_3(t,r,\theta,\phi)}{-2\,M+r},$$

$$G_{ab}[3,1,1,2,4] = \frac{M\,F_{34}(t,r,\theta,\phi) + 8\,r^3\,\sin(\theta)\,\omega_1(t,r,\theta,\phi)}{2\,\sqrt{1-\frac{2\,M}{r}}\,r^2},$$

$$G_{ab}[3,1,1,2,3] = \frac{-F_{11}(t,r,\theta,\phi) - \frac{M\,F_{33}(t,r,\theta,\phi)}{\sqrt{1-\frac{2\,M}{r}}\,r^2}}{2},$$

$$G_{ab}[3,2,2,3,4] = \frac{\sqrt{1-\frac{2\,M}{r}}\,r\,\sin(\theta)\,F_{42}(t,r,\theta,\phi)}{2},$$

$$G_{ab}[3,2,1,3,4] = \frac{(-2\,M+r)\,F_{14}(t,r,\theta,\phi) - \sqrt{1-\frac{2\,M}{r}}\,r^2\,\sin(\theta)\,\left(F_{41}(t,r,\theta,\phi) + 8\,\omega_3(t,r,\theta,\phi)\right)}{2\,r},$$

$$G_{ab}[3,2,1,2,4] = 4\,\sqrt{1-\frac{2\,M}{r}}\,r\,\sin(\theta)\,\omega_2(t,r,\theta,\phi),$$

$$G_{ab}[3,2,1,2,3] = \frac{-\left(\left(1-\frac{2\,M}{r}\right)\,F_{12}(t,r,\theta,\phi)\right)}{2},$$

$$G_{ab}[3,4,2,3,4] = \frac{\frac{\sqrt{1-\frac{2\,M}{r}}\,r\,\cos(\theta)\,F_{23}(t,r,\theta,\phi)}{2\,M-r} + \sin(\theta)\,F_{33}(t,r,\theta,\phi) + F_{44}(t,r,\theta,\phi)}{2},$$

$$G_{ab}[3,4,1,3,4] = \frac{\sqrt{1-\frac{2\,M}{r}}\,\cos(\theta)\,F_{13}(t,r,\theta,\phi)}{2},$$

$$G_{ab}[3,4,1,2,4] = \frac{1}{2\,\sqrt{1-\frac{2\,M}{r}}\,r}\,(2\,M-r)\,\cos(\theta)\,F_{12}(t,r,\theta,\phi)$$

$$+r\,\left(\cos(\theta)\,F_{21}(t,r,\theta,\phi) + \sqrt{1-\frac{2\,M}{r}}\,\left(-\left(\sin(\theta)\,F_{31}(t,r,\theta,\phi)\right) + 8\,\omega_4(t,r,\theta,\phi)\right)\right),$$

$$G_{ab}[3,4,1,2,3] = \frac{-F_{41}(t,r,\theta,\phi) - 8\,\omega_3(t,r,\theta,\phi)}{2},$$

$$G_{ab}[4,1,2,3,4] = \frac{r\,\cos(\theta)\,F_{12}(t,r,\theta,\phi) - \sin(\theta)\,F_{13}(t,r,\theta,\phi) - \frac{8\,\sqrt{1-\frac{2\,M}{r}}\,r^2\,\omega_4(t,r,\theta,\phi)}{-2\,M+r}}{2},$$

$$G_{ab}[4,1,1,3,4] = \frac{-\left(r\,\cos(\theta)\,F_{11}(t,r,\theta,\phi)\right)}{2},$$

$$G_{ab}[4,1,1,2,4] = \frac{\sin(\theta)\,F_{11}(t,r,\theta,\phi) + \frac{M\,F_{44}(t,r,\theta,\phi)}{\sqrt{1-\frac{2\,M}{r}}\,r^2}}{2},$$

$$G_{ab}[4,1,1,2,3] = \frac{-\left(M\,F_{43}(t,r,\theta,\phi)\right) + 8\,r^3\,\omega_1(t,r,\theta,\phi)}{2\,\sqrt{1-\frac{2\,M}{r}}\,r^2},$$

$$G_{ab}[4,2,2,3,4] = \frac{r\,\left(\cos(\theta)\,F_{22}(t,r,\theta,\phi) - \sqrt{1-\frac{2\,M}{r}}\,\sin(\theta)\,F_{32}(t,r,\theta,\phi)\right)}{2},$$

$$G_{ab}[4,2,1,3,4] = \frac{1}{2\,r}\,(2\,M-r)\,\sin(\theta)\,F_{13}(t,r,\theta,\phi)$$

$$+r^2\,\left(-\left(\cos(\theta)\,F_{21}(t,r,\theta,\phi)\right) + \sqrt{1-\frac{2\,M}{r}}\,\left(\sin(\theta)\,F_{31}(t,r,\theta,\phi) - 8\,\omega_4(t,r,\theta,\phi)\right)\right),$$

$$
\begin{aligned}
G_{ab}[4,2,1,2,4] &= \frac{\left(1-\frac{2\,M}{r}\right)\,\sin(\theta)\,F_{12}(t,r,\theta,\phi)}{2},\\
G_{ab}[4,2,1,2,3] &= 4\,\sqrt{1-\frac{2\,M}{r}}\,r\,\omega_2(t,r,\theta,\phi),\\
G_{ab}[4,3,2,3,4] &= \frac{\frac{\cos(\theta)\,F_{23}(t,r,\theta,\phi)}{\sqrt{1-\frac{2\,M}{r}}} - \sin(\theta)\,F_{33}(t,r,\theta,\phi) - F_{44}(t,r,\theta,\phi)}{2},\\
G_{ab}[4,3,1,3,4] &= \frac{-\left(\sqrt{1-\frac{2\,M}{r}}\,\cos(\theta)\,F_{13}(t,r,\theta,\phi)\right)}{2},\\
G_{ab}[4,3,1,2,4] &= \frac{1}{2\,\sqrt{1-\frac{2\,M}{r}}\,r}\,(-2\,M+r)\,\cos(\theta)\,F_{12}(t,r,\theta,\phi)\\
&+r\,\left(-\left(\cos(\theta)\,F_{21}(t,r,\theta,\phi)\right) + \sqrt{1-\frac{2\,M}{r}}\,\left(\sin(\theta)\,F_{31}(t,r,\theta,\phi) - 8\,\omega_4(t,r,\theta,\phi)\right)\right),\\
G_{ab}[4,3,1,2,3] &= \frac{F_{41}(t,r,\theta,\phi) + 8\,\omega_3(t,r,\theta,\phi)}{2}.
\end{aligned}
\tag{A4}
$$

We note now that $F_{21} = F^2_1, F_{31} = F^3_1$ can be the only non-vanishing components compatible with (A4), i.e. $\omega_\mu = 0$ and $F_{ij} = 0$ with $i \neq 2,3$ and $j \neq 1$. To satisfy the field equations one should have $F_{21} = F_{31} = 0$, because (cf. Sec. VI)

$$
\begin{aligned}
K_{1,1,3,4} &= \frac{-8\,M\,\sin(\theta)\,F_{21}(t,r,\theta,\phi)}{r},\\
K_{1,1,2,4} &= \frac{4\,M\,\sin(\theta)\,F_{31}(t,r,\theta,\phi)}{\sqrt{1-\frac{2\,M}{r}}\,r^2},
\end{aligned}
\tag{A5}
$$

but if we do so we fail to satisfy that part of the torsion constraint that involves the component along the Minkowski metric η_{ab}, i.e. $\widetilde{T}^a \wedge V^b$, which requires having at least $F_{31} = F^3_1 \neq 0$. For example, one of the components is given by

$$
\begin{aligned}
&8\left(-\left(\Lambda^{23}\,\cos(\theta)\right) + \frac{6\,M\,\Lambda^{23}\,\cos(\theta)}{r} + r\,({F_1}^3)^{(0,0,0,1)}(t,r,\theta,\phi)\right.\\
&-\sqrt{1-\frac{2\,M}{r}}\,(F_3)^{(0,0,0,1)}(t,r,\theta,\phi) - r\,\sin(\theta)\,({F_1}^4)^{(0,0,1,0)}(t,r,\theta,\phi)\\
&+\sqrt{1-\frac{2\,M}{r}}\,(F_4)^{(0,0,1,0)}(t,r,\theta,\phi) + r\,\sin(\theta)\,({F_3}^4)^{(1,0,0,0)}(t,r,\theta,\phi)\\
&\left.-\;r\,({F_4}^3)^{(1,0,0,0)}(t,r,\theta,\phi)\right) = 0,
\end{aligned}
\tag{A6}
$$

where the four upstairs indices separated by commas denote how many derivatives are taken with respect to t, r, θ, ϕ, respectively.

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